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Relations between two matrices: Fibonacci-Pascal matrix and inverse FFS polynomials matrix

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Abstract. In this article, we prove basic properties such as addition and multiplication of the Fibonacci-Frobenius-sigmoid (FFS) polynomials. we prove basic properties such as addition and multiplication of the Fibonacci-Frobenius-sigmoid (FFS) polynomials. After factorizing the FFS polynomials matrix by the Fibonacci matrix, a certain relation is derived between the inverse FFS polynomials matrix and the Fibonacci-Pascal matrix.

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1. Introduction

First introduced by the Indian mathematician Pingala, the Fibonacci sequence is a sequence in which every number is the sum of the two preceding ones. Fibonacci sequence is closely related to the golden ratio (see [6, 9]). Binet's formula is easily mentioned as the formula expresses the n-th Fibonacci number by using the golden ratio and its conjugate value. As n increases, the ratio of two consecutive Fibonacci numbers converges to the golden ratio (see [9]).

Lucas numbers also share the same recursive relationship with Fibonacci numbers in the sense that each term is the sum of the two previous terms. However, Lucas numbers start with different values compared to Fibonacci numbers (see [1]).

Based on the basic concepts of the Fibonacci sequence and its coefficients, we define the Fibonacci exponential function.

Definition 1.1 [6, 7]. Let $0 \leq n \leq m$ where n and m are non-negative integers. Fibonacci coefficients are defined as

$$\binom{m}{n}_F = \frac{F_m!}{F_{m-n}!F_n!},$$

where $F_m! = F_m F_{m-1} F_{m-2} \cdots F_1$ and $F_0! = 1$.

We note that $\binom{m}{0}_F = 1$ and $\binom{m}{n}_F = 0$ for $m < n$.

Now, consider the exponential function containing Fibonacci numbers as the denominator.

Definition 1.2 [6, 7]. Let $t \in \mathbb{R}$ and n be a non-negative integer. The Fibonacci exponential function e_F^t is

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!}.$$

Unique Pascal matrices can be defined depending on what type of binary coefficient is used (see [2, 5]).

For example, Pascal matrix can be uniquely defined by using a certain coefficient called Fontené-Ward generalized binomial coefficients (see [5]). However, continuing on with the previous topic, we consider Pascal matrices with Fibonomial coefficients.

Definition 1.3 [2, 7]. The Fibonacci-Pascal matrix $\boldsymbol{\mathcal{P}}_n[x] = (p_n(x;i,j))$ is defined as

$$p_n(x;i,j) = \begin{cases} \binom{i}{j}_F x^{i-j}, & if\ i \geq j, \\ 0, & otherwise. \end{cases}$$

Definition 1.4 [7]. For $n \geq 2$, the inverse of the Fibonacci-Pascal matrix $\boldsymbol{\mathcal{V}}_n[x] = (v_n(x; i, j))$ is defined as

$$v_n(x; i, j) = \begin{cases} b_{i-j+1} \binom{i}{j}_F x^{i-j}, & if\ i \geq j, \\ 0, & otherwise, \end{cases}$$

where $b_1 = 1$ and $b_n = -\sum_{k=1}^{n-1} b_k \binom{n}{k}_F$.

Sigmoid function, also known as the logistic curve, is a differentiable, bounded, real function defined for all real input values and has a non-negative derivative at each point with one inflection point. Based on the shape of the logistic curve, it is also known as the S-shaped sigmoid curve (see [8]).

In artificial neural networks, the term is used as the logistic function and the non-smooth functions, also known as hard sigmoids are used for efficiency as well (see [12]).

Generally, a sigmoid function is used when there is a shortcoming in a specific mathematical model. For example, the van Genuchten-Gupta model is based on the S-curve and used to model the relation between the speed of growth in wheat fields and soil salinity (see [10]).

Definition 1.5 [3]. Sigmoid numbers and polynomials are defined respectively as

$$\sum_{n=0}^{\infty} S_n \frac{t^n}{n!} = \frac{1}{e^{-t}+1},$$
$$\sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} = \frac{1}{e^{-t}+1} e^{tx}.$$

Definition 1.6 [4]. Let n be a non-negative integer. Then, Fibonacci sigmoid polynomials is

$$\sum_{n=0}^{\infty} S_{n,F}(x)\frac{t^n}{F_n!} = \frac{1}{e_F^{-t}+1}e_F^{tx}.$$

For $x = 0$ in Definition 1.8, we note

$$\sum_{n=0}^{\infty} S_{n,F}\frac{t^n}{F_n!} = \frac{1}{e^{-t}+1}.$$

where we call $S_{n,F}$ the Fibonacci sigmoid numbers.

Definition 1.7. Let $u \in \mathbb{C} \setminus \{1\}$ and n be a non-negative integer. The generating function of the FFS polynomials $S_{n,F}(x;u)$ is defined as follows:

$$\frac{1-u}{2(e_F^{-t}-u)}e_F^{tx} = \sum_{n=0}^{\infty} S_{n,F}(x;u)\frac{t^n}{F_n!}.$$

For $x = 0$, we note that

$$\sum_{n=0}^{\infty} S_{n,F}(u)\frac{t^n}{F_n!} = \frac{1-u}{2(e_F^{-t}-u)}.$$

Here, $S_{n,F}(u)$ is the FFS number.

To sum up, we introduced the definition of the Fibonacci sequence and its coefficients. Implementing this concept into the exponential function, we defined the Fibonacci exponential function. Then, we introduced the Fibonacci-Pascal matrix and its inverse.

Finally, we summarized the definition and applications of the sigmoid functions and defined the Fibonacci sigmoid polynomials via their generating function.

This paper is roughly outlined as follows: In Section 2, basic operations such as addition and multiplication of FFS polynomials are proved. In Section 3, we factorize the FFS polynomials matrix and derive a relation to the Fibonacci-Pascal matrix.

2. Basic operations on FFS polynomials

In this section, we desire to find a relation between the FFS polynomials and numbers. Based on the additivity of the Fibonacci exponential function, we also prove the additivity and symmetricity property of the FFS polynomials and its numbers.

Theorem 2.1. *Let k be a non-negative integer. Then, we have*

$$S_{n,F}(x;u) = \sum_{k=0}^{n} \binom{n}{k}_F S_{n-k,F}(u) x^k.$$

Proof. To find a relation between FFS numbers and polynomials, we use the generating function as the following:

$$\begin{aligned}\sum_{n=0}^{\infty} S_{n,F}(x;u)\frac{t^n}{F_n!} &= \frac{1-u}{2(e_F^{-t}-u)}e_F^{tx} \\ &= \sum_{n=0}^{\infty} S_{n,F}(u)\frac{t^n}{F_n!}\sum_{n=0}^{\infty} x^n \frac{t^n}{F_n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F S_{n-k,F}(u)x^k\right)\frac{t^n}{F_n!}.\end{aligned} \tag{2.1}$$

Comparing both sides of Equation (2.1), we get the given result. □

Lemma 2.2. *The following holds for the Fibonacci exponential function for all $t, s \in \mathbb{R}$:*

$$e_F^{t+s} = e_F^t e_F^s$$

Proof. For $t, s \in \mathbb{R}$, we have

$$\begin{aligned} e_F^t e_F^s &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{t_F^{n-k}}{F_{n-k}!}\frac{s_F^k}{F_k!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F \frac{t_F^{n-k}s_F^k}{F_n!} \\ &= \sum_{n=0}^{\infty}\frac{(t+s)_F^n}{F_n!} = e_F^{t+s}.\end{aligned} \tag{2.2}$$

□

Additivity property of the following function is a key ingredient used for some of the central results in upcoming results.

Now, we observe the additivity within the FFS polynomials.

Theorem 2.3. *Let $x, y \in \mathbb{C}$. Then, the following equation holds:*

$$S_{n,F}(x+y;u) = \sum_{k=0}^{n} \binom{n}{k}_F S_{n-k,F}(x;u)y^k.$$

Proof. Using Definition 1.7, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} S_{n,F}(x+y;u)\frac{t^n}{F_n!} &= \frac{1-u}{2(e_F^{-t}-u)}e_F^{t(x+y)} \\
&= \sum_{n=0}^{\infty} S_{n,F}(x;u)\frac{t^n}{F_n!}\sum_{n=0}^{\infty} y^n \frac{t^n}{F_n!} \qquad (2.3)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F S_{n-k,F}(x;u)y^k\right)\frac{t^n}{F_n!}.
\end{aligned}$$

The additivity of e_F^{x+y} preserves by Lemma 2.2. Comparing the left and right hand side of (2.3), we derive the stated result. □

Corollary 2.4. *Setting $y = 1$ in Theorem 2.3, it holds that*

$$S_{n,F}(x+1;u) = \sum_{k=0}^{n} \binom{n}{k}_F S_{n-k,F}(x;u).$$

On top of the additivity property, we also consider the symmetric relations of the FFS polynomials and numbers.

Theorem 2.5. *Let $\alpha, \beta \in \mathbb{R}$. Then, we have the following basic symmetric relation:*

$$\begin{aligned}
&\sum_{k=0}^{n} \alpha^{n-k}\beta^k S_{n-k,F}(\alpha^{-1}x;u)S_{k,F}(\beta^{-1}y;u) \\
&= \sum_{k=0}^{n} \beta^{n-k}\alpha^k S_{n-k,F}(\beta^{-1}x;u)S_{k,F}(\alpha^{-1}y;u).
\end{aligned}$$

Proof. We consider form A as

$$A := \frac{(1-u)^2 e_F^{t(x+y)}}{4(e_F^{-\alpha t}-u)(e_F^{-\beta t}-u)}. \tag{2.4}$$

Organizing the numerator of A, we have

$$\begin{aligned} A &:= \frac{(1-u)}{2(e_F^{-\alpha t}-u)} e_F^{tx} \frac{(1-u)}{2(e_F^{-\beta t}-u)} e_F^{ty} \\ &= \sum_{n=0}^{\infty} S_n(\alpha^{-1}x;u)\frac{(\alpha t)^n}{F_n!} \sum_{n=0}^{\infty} S_n(\beta^{-1}y;u)\frac{(\beta t)^n}{F_n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha^{n-k}\beta^k S_{n-k}(\alpha^{-1}x;u) S_k(\beta^{-1}y;u) \right) \frac{t^n}{F_n!}. \end{aligned} \tag{2.5}$$

Also, by switching the order of the split numerator of A, the following also holds:

$$\begin{aligned} A &:= \frac{(1-u)}{2(e_F^{-\beta t}-u)} e_F^{tx} \frac{(1-u)}{2(e_F^{-\alpha t}-u)} e_F^{ty} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \beta^{n-k}\alpha^k S_{n-k}(\beta^{-1}x;u) S_k(\alpha^{-1}y;u) \right) \frac{t^n}{F_n!}. \end{aligned} \tag{2.6}$$

Hence, we can obtain the basic symmetric relation within the FFS polynomials by comparing (2.5)and (2.6). □

Corollary 2.6. *Fix $\alpha = 1$ in Theorem 2.5. Then, the symmetric relation within the FFS polynomials with respect to β becomes*

$$\begin{aligned} &\sum_{k=0}^{n} \beta^k S_{n-k,F}(x;u) S_{k,F}(\beta^{-1}y;u) \\ &= \sum_{k=0}^{n} \beta^{n-k} S_{n-k,F}(\beta^{-1}x;u) S_{k,F}(y;u). \end{aligned}$$

Corollary 2.7. *Let $x = 0$ in Theorem 2.5. Then, the symmetric relation between FFS numbers would be*

$$\begin{aligned} &\sum_{k=0}^{n} \alpha^{n-k}\beta^k S_{n-k,F}(u) S_{k,F}(\beta^{-1}y;u) \\ &= \sum_{k=0}^{n} \beta^{n-k}\alpha^k S_{n-k,F}(u) S_{k,F}(\alpha^{-1}y;u). \end{aligned}$$

3. Relations between FFS polynomials matrix and Fibonacci-Pascal matrix

In this section, we initially define the inverse FFS polynomials matrix. Intuitively, the factorization of the FFS polynomials matrix via the Fibonacci matrix is shown. Also, we obtain an equation that represents the relation between the FFS polynomials matrix and the Fibonacci-Pascal matrix. Ultimately, from this relation, we can easily derive the relation between the FFS numbers matrix and the Fibonacci-Pascal matrix.

Proposition 3.1. *The inverse of the* $(n+1)\times(n+1)$ *FFS polynomials matrix* $\boldsymbol{\mathcal{D}}_{n,F}(x;u)=[d_{ij,F}]$ *is defined as*

$$d_{ij,F}=\begin{cases}\binom{i}{j}_F\sum_{k=0}^{i-j}\binom{i-j}{k}_F\dfrac{1}{F_{i-k-j+1}}b_{k+1}x^{k-j}, & if \quad i\geq j,\\ 0, & otherwise.\end{cases}$$

Corollary 3.2. *The FFS polynomials matrix* $\boldsymbol{\mathcal{S}}_{n,F}(x;u)$ *can be factorized in terms of Fibonacci matrix* $\boldsymbol{\mathcal{F}}_n$ *as follows:*

$$\boldsymbol{\mathcal{S}}_{n,F}(x;u)=\boldsymbol{\mathcal{F}}_n\boldsymbol{\mathcal{M}}_{n,F}(x;u).$$

If $x=0$*, the following holds:*

$$\boldsymbol{\mathcal{F}}_n\boldsymbol{\mathcal{M}}_{n,F}(u)=\boldsymbol{\mathcal{S}}_{n,F}(u).$$

Based on Corollary 3.2, we derive an example of $\boldsymbol{\mathcal{M}}_{n,F}(u)$ when $n=3$.

Example 3.3. Consider $n=3$. For the FFS polynomials matrix, we have

$$\boldsymbol{\mathcal{F}}_3\boldsymbol{\mathcal{M}}_{3,F}(x;u)$$
$$=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&0&0\\1&2&2&0\end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2}x + \frac{u}{2(1-u)} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}x^2 - \frac{3-2u}{2(1-u)}x - \frac{2+u}{2(1-u)} - \frac{1}{2(1-u)^2} & \frac{1}{2}x + \frac{2u-1}{2(1-u)} & \frac{1}{2} & 0 \\ \begin{array}{c} \frac{1}{3}x^3 + \frac{u}{1-u}x^2 - \left(\frac{5-2u}{2(1-u)} + \frac{1}{2(1-u)^2}\right)x \\ -\frac{2+2u}{1-u} - \frac{1}{(1-u)^2} - \frac{1}{(1-u)^3} - \frac{1}{2} \end{array} & x^2 - \frac{2-u}{1-u}x - \frac{1+u}{1-u} - \frac{1}{(1-u)^2} & x + \frac{u}{1-u} & \frac{1}{2} \end{bmatrix}$$
$$= \boldsymbol{\mathcal{S}}_{n,F}(x;u).$$

Hence, we obtained the factorization of FFS polynomials matrix via the Fibonacci matrix and derived an example of the factorization process.

Proposition 3.4. *Let $S_{k,F}(u)$ be the FFS number. Then, consider the following statement:*

$$\sum_{k=0}^{n} \frac{1}{F_{n-k+1}} \binom{n}{k}_F S_{k,F}(u) = F_n! \delta_{n,0}$$

where $\delta_{n,0}$ is the Kronecker delta function.

Theorem 3.5. *The following relation between the inverse FFS polynomials matrix and the Fibonacci-Pascal matrix holds:*

$$\boldsymbol{\mathcal{D}}_{n,F}(x;u) = \frac{1}{2}\left(\boldsymbol{\mathcal{P}}_{n+1}[x] + \boldsymbol{\mathcal{I}}_{n+1}\right).$$

Proof. Let $\sum_{k=0}^{n} \binom{n}{k}_F S_{n-k,F}(x;u)x^k + S_{n,F}(x;u) = 2\delta_{n,i-j}$.

Then, the following equation is verified by substituting $i-j$ instead of i for the values of k and using Proposition 3.4.

$$\begin{aligned} &\left(\boldsymbol{\mathcal{S}}_{n,F}(x;u)\left(\boldsymbol{\mathcal{P}}_{n+1}[x] + \boldsymbol{\mathcal{I}}_{n+1}\right)\right)_{ij} \\ &= \sum_{k=j}^{i} \binom{i}{k}_F S_{i-k,F}(x;u) \binom{k}{j}_F x^{k-j} + \binom{i}{j}_F S_{i-j,F}(x;u) \\ &= \binom{i}{j}_F \sum_{k=j}^{i} \binom{i-j}{k-j}_F S_{i-k,F}(x;u)x^{k-j} + \binom{i}{j}_F S_{i-j,F}(x;u) \end{aligned}$$

$$
\begin{aligned}
&= \binom{i}{j}_F \left[\sum_{k=0}^{i-j} \binom{i-j}{k}_F S_{i-j-k,F}(x;u)x^k + S_{i-j,F}(x;u) \right] \\
&= \binom{i}{j}_F 2\delta_{0,i-j}.
\end{aligned} \tag{3.1}
$$

For $i = j$, $\binom{i}{j}_F \delta_{0,i-j} = 1$ holds.

On the other hand, $\binom{i}{j}_F \delta_{0,i-j} = 0$ when $i \neq j$. Hence, the inverse matrix is obtained. □

Based on the Kronecker delta function previously defined for the FFS number, we now define the Kronecker delta function for the FFS number matrix.

Proposition 3.6. *Let $\boldsymbol{\mathcal{S}}_{n,F}(u)$ be the FFS number matrix. Then, it holds that*

$$\sum_{k=0}^{n} \binom{n}{k}_F \boldsymbol{\mathcal{S}}_{n-k,F}(u) + \boldsymbol{\mathcal{S}}_{n,F}(u) = 2\delta_{0,n}.$$

Theorem 3.7. *Let $\boldsymbol{\mathcal{P}}_{n+1} = [p_{ij}]$ be the $(n+1) \times (n+1)$ Fibonacci-Pascal matrix and $\boldsymbol{\mathcal{I}}_{n+1}$ be the identity matrix. Also, let $\boldsymbol{\mathcal{S}}_{n,F}(u)$ be the FFS numbers matrix. Then, the following equation holds:*

$$\frac{1}{2}\left(\boldsymbol{\mathcal{P}}_{n+1} + \boldsymbol{\mathcal{I}}_{n+1}\right) = \boldsymbol{\mathcal{S}}_{n,F}(u)^{-1}.$$

Proof. Consider the above proposition and recall the calculation of the proof of Theorem 3.5. Then, we have

$$
\begin{aligned}
&\left(\boldsymbol{\mathcal{S}}_{n,F}(u) \frac{1}{2}\left(\boldsymbol{\mathcal{P}}_{n+1} + \boldsymbol{\mathcal{I}}_{n+1}\right) \right)_{ij} \\
&= \frac{1}{2}\left(\boldsymbol{\mathcal{S}}_{n,F}(u)\boldsymbol{\mathcal{P}}_{n+1} + \boldsymbol{\mathcal{S}}_{n,F}(u)\right)_{ij} \\
&= \sum_{k=j}^{i} \binom{i}{k}_F \boldsymbol{\mathcal{S}}_{i-k,F} \frac{1}{2} \binom{k}{j}_F + \frac{1}{2}\binom{i}{j}_F \boldsymbol{\mathcal{S}}_{i-j,F}(u)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\binom{i}{j}_F \sum_{k=j}^{i} \binom{i-j}{k-j}_F \boldsymbol{\mathcal{S}}_{i-k,F}(u) + \frac{1}{2}\binom{i}{j}_F \boldsymbol{\mathcal{S}}_{i-j,F}(u) \\
&= \frac{1}{2}\binom{i}{j}_F \left[\sum_{k=j}^{i} \binom{i-j}{k-j}_F \boldsymbol{\mathcal{S}}_{i-k,F}(u) + \boldsymbol{\mathcal{S}}_{i-j,F}(u)\right] \qquad (3.2) \\
&= \frac{1}{2}\binom{i}{j}_F 2\delta_{0,i-j} = \binom{i}{j}_F \delta_{0,i-j}.
\end{aligned}
$$

Hence, for $i = j$, $\binom{i}{j}_F \delta_{0,i-j} = 1$ holds. Also, for $i \neq j$, $\binom{i}{j}_F \delta_{0,i-j} = 0$. Therefore, multiplication of $\boldsymbol{\mathcal{S}}_{n,F}(u)$ and $\frac{1}{2}(\boldsymbol{\mathcal{P}}_{n+1} + \boldsymbol{\mathcal{I}}_{n+1})$ becomes an identity matrix. □

4. Conclusion

After observing the Fibonacci exponential function and the generating function of the FFS polynomials, we derived basic properties for the FFS polynomials. Based on the form of the Fibonacci-Pascal matrix and the inverse FFS polynomials matrix, we were able to prove the related form between them using the identity matrix and the Kronecker-delta function.

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